A MODIFIED VARIATIONAL PRINCIPLE IN RELATIVISTIC HYDRODYNAMICS
I. Commutativity and Noncommutativity of the Variation Operator with the Partial Derivative

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Abstract

A model is proposed, according to which the metric tensor field in the standard gravitational Lagrangian is decomposed into a projection (generally - with a non-zero covariant derivative) tensor field, orthogonal to an arbitrary 4-vector field and a tensor part along the same vector field. A theorem has been used, according to which the variation and the partial derivative, when applied to a tensor field, commute with each other if and only if the tensor field and its variation have zero covariant derivatives, provided also the connection variation is zero. Since the projection field obviously does not fulfill the above requirements of the "commutation" theorem, the exact expression for the (non-zero) commutator of the variation and the partial derivative, applied to the projection tensor field, can be found from a set of the three defining equations. The above method will be used to construct a modified variational approach in relativistic hydrodynamics, based on variation of the vector field and the projection field, the last one thus accounting for the influence of the reference system of matter (characterized by the 4-vector) on the gravitational field.

I. INTRODUCTION.

The application of variational formalism in relativistic hydrodynamics has always been a key problem for the adequate description of the combined system of gravitational field and matter, understood in the sense of a *perfect fluid*. In the notion of Dewitt [1], the perfect fluid is a "stiff elastic" medium. The most important physical quantities, allowing us to view the perfect fluid (or matter) as a macro-body are the *mass-density* ρ_o, *the internal energy density* w_o and *the macrovelocity* u, the latter one making possible to characterize the position of each particle of the ideal fluid. For all these macrovariables the the corresponding conservation laws of *mass* and

internal energy (or entropy) can be defined. An important basic point in the starting investigations [2,3] of relativistic hydrodynamics is the following fact. Suppose the Einstein's equation is written as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \equiv T_{\mu\nu}, \tag{1}$$

with the source term $T_{\mu\nu}$ in the right-hand side - *the stress-energy tensor* of matter (in the case- the ideal fluid)

$$T_{\mu\nu} \equiv \rho_0 u^{\mu} u^{\nu} - p g^{\mu\nu}. \tag{2}$$

Then it can be proved that (1) and (2) are in fact the Euler's equations due to the variation of $g_{\mu\nu}$ in the action functional

$$I = I_{matter} + I_{fluid} = \int \left[R - 2\rho_o (c^2 + H_o + \frac{1}{2} \mu g_{\mu\nu} u^{\mu} u^{\nu}) \sqrt{-g} d^4 x \right], \tag{3}$$

where R is the scalar curvature of the gravitational field, μ is a constant and

$$H_o = w_o - T S^o \tag{4}$$

is the *Helmholz free energy,* S_o is the *rest entropy* of the fluid. However, this variational formalism suffers a defect - it is based on the application of definite constraints, added in the standard Gravitational Lagrangian in such a way that upon variation the *mass conservation equation*

$$(\rho_o u^{\nu})_{|\nu} \equiv 0 \tag{5}$$

($_{|\nu}$ means covariant differentiation)is fullfilled (u^k is a unit vector, such that $g_{\mu\nu} u^{\mu} u^{\nu} \equiv 1$).In fact, the addition of such constraints of a definite form is dictated by the necessity to sideway the basic difference between general relativity and hydrodynamics - while in general relativity all conservation laws are found on the base of a *variational principle*, in hydrodynamics these conservation laws are of a "continuim" nature, in the sense that they are a consequence of the conservation of mass and energy. Therefore, the hydrodynamical equations, *are not derived on the base of a variational principle.* Note that in this formalism the equations of motion of the fluid are nothing else but the equations of *energy-momentum conservation*

$$T^{\mu\nu}_{|\nu} \equiv (R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R)_{|\nu}. \tag{6}$$

Also, it is evident that *the velocity field does not enter the gravitational part of the Lagrangian.* Due to this reason the velocity field is not subjected to a "true", physically reasonable variational principle.

In this paper another point of view shall be assumed - *if $u^{\alpha}(x)$ is the attached to each space-time point of the fluid (medium) velocity four-vector, then all physical vector and tensor variables in the theory (for example - the metric tensor $g_{\mu\nu}$) should by projected either along this velocity field,or along the perpendicular direction.* For example, if $e = u^k u_k$ is the velocity length vector, then the tensor

$$p_{\mu\nu} \equiv g_{\mu\nu} - \frac{1}{e} u_{\mu} u_{\nu} \tag{7}$$

is called *a projection tensor.* Provided the vector field indices are raised and lowered with the metric tensor, it can easily be checked that

$$u^{\nu} p_{\mu\nu} \equiv 0. \tag{8}$$

Therefore, at each spacetime point and at each instant of time the vector field u is orthogonal to the projection tensor. Three other facts are very important:

1. u is an *arbitrary, not normalized, not time-like or space like vector field.* In this sense, the transformation (7) can be viewed as a more general one in comparison with the ADM approach [4], where u was assumed to be a timelike vector and the projection tensor $p_{\mu\nu}$-spacelike, and also in comparison with the recently used in [5, 6] decomposition

$$\overline{g_{ij}} = g_{ij} + 2H l_i l_j, \tag{9}$$

where g_{ij}, named a seed metric, was assumed to be a metric field (taken to be conformally flat), and $l_i -$ a null vector field ($H = const.$).

2. If $g_{\mu\nu}$is a metric tensor and $g^{\mu\nu}$ is its inverse, then $p_{\mu\nu}$ *is not a metric tensor*, since from the defining equation (7) it easily follows

$$p_{\mu\nu} p^{\nu\alpha} \equiv g_{\mu}^{\alpha} - \frac{1}{e} u^{\alpha} u_{\mu}, \tag{10}$$

where

$$g_{\mu}^{\alpha} \equiv \delta_{\mu}^{\alpha} \equiv \{1 if \alpha = \mu and 0 if \alpha \neq \mu\}. \tag{11}$$

3. If $g_{\mu\nu}$ has a zero-covariant derivative with respect to the given connection $\Gamma_{\mu\nu}^{\alpha}$, i.e.$\nabla_{\alpha} g_{\mu\nu} \equiv 0$ ($\nabla_{\alpha}-$covariant derivative), then $p_{\mu\nu}$ does not have this property, because

$$\nabla_\alpha p_{\mu\nu} = -\nabla_\alpha(\frac{1}{e}u_\mu u_\nu). \tag{12}$$

Evidently, this covariant derivative will be zero if $\nabla_\alpha u_\mu = 0$, but the last would mean that a special kind of transport of the vector field u has been assumed. In fact, it turns out that the non-zero covariant derivative of the tensor field $p_{\mu\nu}$ has profound consequences for the variational formalism (called a *projection variational formalism*), which will be based on variations of the projection field $p_{\mu\nu}$ and the vector field u and their (partial) derivatives.This will be further investigated.

The present paper is organized as follows: In Section II a proof will be given (see [7] for the original version, also [8]) that the (form) variation commutes with the partial derivative in the case of zero-covariant derivative of the tensor field, provided also that the (form) variation of this field has also zero-covariant dereivative and the (form) variation of the connection is zero. In Section III a simple exaple will be given that in perturbative gravity theory the (form)variation and the partial differentiation *do not commute.* And as far as the consideration of the projection field as a dynamical field variable is concerned, it is clear that *the projection variational formalism strictly does not allow the implementation of such a "commutativity" assumption due to the last property 3 and also to the fact that the projection connection variation is different from zero (even in case the initial connection variation is zero)***.** Due to this reason, in Section III the exact commutation relation between the variation and the partial derivative, applied to the projection tensor, will be found.

II. WHEN DO THE VARIATION AND THE PARTIAL DERIVATIVE COMMUTE?

In order to have an adequate understanding about the projection variational formalism, one should first try to understand whether the commutation property between the variation and the partial derivative, applied in the theory of gravity for the derivation of the boundary terms for example, is such a common property, which is suppposed to be applied without any restrictions and assumptions.

The proposition below, originally proved in [7],will show that in fact the commutation property places some restrictions and therefore it *cannot* always be applied. In fact, as noted in [8], in differentiable manifolds *without affine*

connections the "commutativity" property is always fulfilled since the functional (form) variation is independent from the change of coordinate "maps". But in differentiable manifolds with affine connections, there is a relation between the functional (form) variation and the covariant differentiation. This point of reasoning will be explained in more details in the Discussion part of the present paper.

PROPOSITION 1 [9]. Let us denote by $P_{\mu\nu\alpha}$ the following expression:

$$P_{\mu\nu\alpha} \equiv \overline{\delta}\partial_\alpha g_{\mu\nu} - \partial_\alpha \overline{\delta} g_{\mu\nu} \equiv \left[\overline{\delta}, \partial_\alpha\right] g_{\mu\nu} \tag{13}$$

and the variation is understood as a *form* variation $\overline{\delta}$, namely

$$\overline{\delta p}_{ij} \equiv p'_{ij}(x) - p_{ij}(x), \tag{14}$$

thus representing the difference between the functional values, taken at one and the same point.

Let us also assume that:

1. The covariant derivative of the background metric tensor in respect to the background connection is zero, i.e. $\mathrm{g}_{\mu\nu|\alpha} = 0$.

2. The covariant derivative of the background metric tensor variation is zero, i. e. $(\overline{\delta} g_{\mu\nu})_{|\alpha} \equiv 0$ and therefore after the variation the metric remains again within the class of Riemannian metrics with zero-covariant derivative.

Then $P_{ijk} \equiv 0$ if and only if $\overline{\delta}\Gamma^\alpha_{\mu\nu} \equiv 0$. In other words, *the variation commutes with the partial derivative if and only if the variation of the background connection is zero.*

Proof: The proof is based essentially on the fact that the variation $\delta g_{\mu\nu}$ is again a second - rank tensor and therefore the standard formulae for its covariant derivative is valid:

$$(\overline{\delta} g_{\mu\nu})_{|\alpha} = \partial_\alpha \overline{\delta} g_{\mu\nu} + \Gamma^\sigma_{\mu\alpha}\overline{\delta} g_{\sigma\nu} + \Gamma^\sigma_{\nu\alpha}\overline{\delta} g_{\sigma\mu}. \tag{15}$$

Using the usual formulae for the covariant derivative $g_{\mu\nu|\alpha}$and afterwards taking its variation, one can obtain:

$$\overline{\delta}(g_{\mu\nu|\alpha}) = \overline{\delta}\partial_\alpha g_{\mu\nu} + \overline{\delta}\Gamma^\sigma_{\mu\alpha} g_{\sigma\nu} + \overline{\delta}\Gamma^\sigma_{\nu\alpha} g_{\sigma\mu} + \Gamma^\sigma_{\mu\alpha}\overline{\delta} g_{\sigma\nu} + \Gamma^\sigma_{\nu\alpha}\overline{\delta} g_{\sigma\mu}. \tag{16}$$

From (15) and (16) it can be obtained:

$$P_{\mu\nu\alpha} = \overline{\delta}(g_{\mu\nu|\alpha}) - (\overline{\delta} g_{\mu\nu})_{|\alpha} - \overline{\delta}\Gamma^\sigma_{\mu\alpha} g_{\sigma\nu} - \overline{\delta}\Gamma^\sigma_{\nu\alpha} g_{\sigma\mu} = \tag{17}$$

$$= -\overline{\delta}\Gamma^{\sigma}_{\mu\alpha} g_{\sigma\nu} - \overline{\delta}\Gamma^{\sigma}_{\nu\alpha} g_{\sigma\mu}, \tag{18}$$

in accordance with the first and the second assumptions of this theorem. From (17) it is clear that if $\overline{\delta}\Gamma^{\sigma}_{\mu\nu} \equiv 0$, then $P_{\mu\nu\alpha} \equiv 0$. Let us now assume that $P_{\mu\nu\alpha} \equiv 0$. Then by cyclic permutation of the indices one can obtain the relation:

$$0 \equiv P_{\mu\nu\alpha} + P_{\nu\alpha\mu} - P_{\alpha\mu\nu} \equiv -2\overline{\delta}\Gamma^{\sigma}_{\mu\alpha} g_{\sigma\nu} \tag{19}$$

and therefore $\overline{\delta}\Gamma^{\sigma}_{\mu\alpha} \equiv 0$. This precludes the proof of the proposition.

It is clear from the above proof that the commutaion property will no longer be valid for Einstein-Cartan theories, where the metric and the connection are being varied independently. It is understood also that the requirement about the zero connection variation is a more stronger one than the requirement about the metric tensor, which means that even within the class of Riemannian space-times the commutativity between the variation and the partial derivative will not always be fulfilled.

PROPOSITION 2. Let us represent the metric tensor g_{im} as

$$g_{im} = p_{im} + \frac{1}{e} u_i u_m \tag{20}$$

and let us also assume that the (form) variation and the partial derivative commute in respect to the *metric* tensor g^{im}, i.e.

$$\left[\overline{\delta}, \partial_j\right] g^{im} \equiv 0. \tag{21}$$

Then this relation, combined with the orthogonality condition

$$u^i p_{ik} \equiv 0 \tag{22}$$

leads to raising and lowering of the indices of the projection tensor with the metric tensor, provided the indices of the vector field are also raised and lowered with the metric tensor. Since this property is generally fulfilled, in spite of any assumptions about commutativity or non-commutativity, the statement here is that in fact the above property in respect to the projection tensor indeed can be proved to follow from the commutativity condition (21). In this sense, it does not provide any new information.

Proof: If (20) is substituted into (21), it can be obtained

$$\left[\overline{\delta}, \partial_j\right] p^{im} + \left[\overline{\delta}, \partial_j\right] \left(\frac{1}{e} u^i u^m\right) \equiv 0. \tag{23}$$

Next, applying the commutator $\left[\overline{\delta}, \partial_j\right]$ to (22), written as $\frac{1}{e}u^i u^m p_{ik} \equiv 0$, it can be derived that

$$\frac{1}{e}u^i u^m \left[\overline{\delta}, \partial_j\right] p_{ik} + p_{ik}\left[\overline{\delta}, \partial_j\right](\frac{1}{e}u^i u^m) \equiv 0. \qquad (24)$$

If the second term in (23) is substituted into (24), the following equation is obtained

$$\frac{1}{e}u^i u^m \left[\overline{\delta}, \partial_j\right] p_{ik} - p_{ik}\left[\overline{\delta}, \partial_j\right] p^{im} \equiv 0. \qquad (25)$$

Taking into account also that

$$\frac{1}{e}u^i u^m \left[\overline{\delta}, \partial_j\right] p_{ik} \equiv g^{im}\left[\overline{\delta}, \partial_j\right] p_{ik} - p^{im}\left[\overline{\delta}, \partial_j\right] p_{ik} \equiv \left[\overline{\delta}, \partial_j\right] p_k^m - p^{im}\left[\overline{\delta}, \partial_j\right] p_{ik} \qquad (26)$$

and making the very important assumption that the indices of the projective tensor are raised and lowered with the metric tensor g_{ik}, equation (25) can be rewritten as

$$\left[\overline{\delta}, \partial_j\right] p_k^m \equiv p^{im}\left[\overline{\delta}, \partial_j\right] p_{ik} + p_{ik}\left[\overline{\delta}, \partial_j\right] p^{im} \equiv \left[\overline{\delta}, \partial_j\right](p^{im}p_{ik}), \qquad (27)$$

and from here

$$p_k^m \equiv p^{mi}p_{ik}. \qquad (28)$$

What is more important to be realized as the main result of this proposition is the following: As a result of the assumption that indices of the projection tensor can be raised and lowered with the given metric, (28) is obtained. *As a matter of fact, however, this equation follows from the defining equations (20) and (22) and also the assumption that the indices of the vector field are raised and lowered also with the metric tensor. In other words, the raising and lowering of projection tensor indices with the metric tensor is not a direct consequence of the commutativity property of the variation and the partial derivative in respect to the metric tensor.* And of course, the last statement will also be true if the vector field and the projection field interchange their places.

III. NONCOMMUTATIVITY OF THE VARIATION WITH THE PARTIAL DERIVATIVE IN PERTURBATIVE GRAVITY - A SIMPLE EXAMPLE

In perturbative gravity the metric tensor $g_{\mu\nu}$ is expanded into power series of the Planck constant $\hbar$:

$$g_{\mu\nu} \equiv g_{\mu\nu}^{(0)} + \hbar g_{\mu\nu}^{(1)} + \hbar^2 g_{\mu\nu}^{(2)} + ... = g_{\mu\nu}^{(0)} + \hbar \overline{\delta} g_{\mu\nu} + \hbar^2 \overline{\delta}^2 g_{\mu\nu} + ... \equiv g_{\mu\nu}^{(0)} + \hbar h_{\mu\nu} + \hbar^2 \overline{\delta} h_{\mu\nu} + ..., \tag{29}$$

where the terms $\overline{\delta} g_{\mu\nu}$, $\overline{\delta}^2 g_{\mu\nu}$ represent fluctuations of the gravitational field to first, second and higher order. Note that while usually the background metric is considered to have zero-covariant derivative, the fluctuations are thought not to have this property. Therefore, after implementation of formulaes (13) and (17), it can be derived

$$\left[\overline{\delta}, \partial_\alpha\right] g_{\mu\nu} \equiv -h_{\mu\nu|\alpha} - \overline{\delta}\Gamma_{\alpha(\mu}^{r} g_{\nu)r}, \tag{30}$$

where $\overline{\delta}(g_{\mu\nu|\alpha}) \equiv 0$ and $\overline{\delta} g_{\mu\nu} \equiv h_{\mu\nu}$.

$$\left[\overline{\delta}, \partial_\alpha\right] h_{\mu\nu} \equiv \overline{\delta}(h_{\mu\nu|\alpha}) - (\overline{\delta} h_{\mu\nu})_{|\alpha} - \overline{\delta}\Gamma_{\alpha(\mu}^{r} h_{\nu)r} \equiv -(\overline{\delta} h_{\mu\nu})_{|\alpha} + \Gamma_{\alpha(\mu}^{r} \overline{\delta} h_{\nu)r}. \tag{31}$$

The two formulaes clearly show that the variation and the partial derivative *do not commute* neither when applied to the background metric, nor to the fluctuating tensor field $h_{\mu\nu}$.

IV. NON-COMMUTATIVITY BETWEEN THE VARIATION AND THE FIRST PARTIAL DERIVATIVE OF THE PROJECTION TENSOR - AN EXACT EXPRESSION FROM THE SET OF THREE DEFINING EQUATIONS.

In the formal sense, a formulae analogous to (31) can also be written, but for the projection field $p_{\mu\nu}$. In this section, however, the exact expression for the commutator shall be derived on the basis of the three equations (constraints), determining the projection metric in respect to the vector field and the initially given metric field, namely:

1. The relation between the covariant and the contravariant components of the projection tensor and the vector field, which is a consequence of the existence of an inverse initial metric tensor $g^{\alpha\beta}$:

$$p_{mk} p^{ik} \equiv \delta_m^i - \frac{1}{e} u^i u_m. \tag{32}$$

2. The relation,expressing the orthogonality of the vector field u in respect to the projection tensor, which for convenience shall be written in the form:

$$\frac{1}{e}u^k u^i p_{km} \equiv 0. \tag{33}$$

3. The relation, derived from the zero-covariant derivative of the initial metric $g_{\alpha\beta}$ (in respect to the initial Christoffel connection $\Gamma^{\gamma}_{\alpha\beta}$) , which can be expressed as an nonlinear equation between the vector field u, the projection tensor field $p_{\alpha\beta}$ and their first partial derivatives

$$\partial_j g^{ki} \equiv -g^{s(k}\Gamma^{i)}_{sj}, \tag{34}$$

or after substituting g^{ik} with $g^{ik} = p^{ik} + \frac{1}{e}u^i u^k$:

$$\partial_j p^{ik} + \partial_j(\frac{1}{e}u^i u^k) = -p^{s(k}\Gamma^{i)}_{sj} - (\frac{1}{e}u^s u^{(k})\Gamma^{i)}_{sj}, \tag{35}$$

where Γ^i_{sj} is the connection of the inilially given metric. In fact, this connection is a function of the projection tensor, the vector field and their first partial derivatives and due to this (35) is a nonlinear equation in respect to these variables. Note also the important fact that (34) is a consequence of the fact that $g_{\alpha\beta}$ has a well defined inverse metric $g^{\alpha\beta}$.

In order to find the exact formulae for the commutator $[\delta, \partial_j]\, p_{ik}$, the operators δ and ∂_j shall be applied in a consequent and afterwards - in an inversed order to all the above given relations (32), (33) and (35). Throughout the whole section it shall be understood that when applied in a consequent order to a given tensor field, first the action of the operator, standing left (i.e. next) to the tensor, shall be assumed. For example, if we perform first partial differentiation ∂_j to equation (32) and afterwards - a variation, we receive

$$p^{ik}(\delta\partial_j p_{km}) + (\partial_j p_{mk})(\delta p^{ki}) + (\delta p_{mk})(\partial_j p^{ki}) + p_{mk}(\delta\partial_j p^{ki}) \equiv -\delta\partial_j(\frac{1}{e}u^i u_m). \tag{36}$$

When applied in an inversed order, i.e. first the variation and then -the partial differentiation, another equation is obtained

$$p^{ik}(\partial_j \delta p_{km}) + (\partial_j p^{ik})(\delta p_{km}) + (\partial_j p_{mk})(\delta p^{ki}) + p_{mk}(\partial_j \delta p^{ki}) \equiv -\partial_j \delta(\frac{1}{e}u^i u_m). \tag{37}$$

From (36) and (37) the formulae for the commutator can be obtained

$$p^{ik}\,[\delta,\partial_j]\,p_{km}+p_{mk}\,[\delta,\partial_j]\,p^{ki}\equiv-\,[\delta,\partial_j]\,(\frac{1}{e}u^iu_m). \tag{38}$$

In the same way, the commutator can be applied to (33) and it can be found

$$\frac{1}{e}u^iu^k\,[\delta,\partial_j]\,p_{km}\equiv-p_{km}\,[\delta,\partial_j]\,(\frac{1}{e}u^ku^i). \tag{39}$$

Having in mind also that indices of the vector field are lowered and lifted with the initial metric and therefore $\frac{1}{e}u^iu_m=\frac{1}{e}u^ig_{mk}u^k$, it can easily be derived that

$$[\delta,\partial_j]\,(\frac{1}{e}u^iu_m)\equiv g_{mk}\,[\delta,\partial_j]\,(\frac{1}{e}u^iu^k)+(\frac{1}{e}u^iu^k)\,[\delta,\partial_j]\,g_{mk}. \tag{40}$$

If the right-hand side of (38) is substituted with the expression from the last equation, the following equation is obtained:

$$p^{ik}\,[\delta,\partial_j]\,p_{km}+p_{mk}\,[\delta,\partial_j]\,p^{ki}\equiv-g_{mk}\,[\delta,\partial_j]\,(\frac{1}{e}u^iu^k)-(\frac{1}{e}u^iu^k)\,[\delta,\partial_j]\,g_{mk}. \tag{41}$$

Now use shall be made of the last relation (34) and the resulting equation (35), following from the zero-covariant derivative of the initial metric. Unlike the previously used equations (32) and (33), (35) includes in itself terms with *partial derivatives* of the vector field and the projection tensor. In other words, *the "source" of noncommutativity is hidden in the fact that the partial differentiation does not enter on an equal footing in all the three equations.* This means that since partial differentiation is already present in (35), only the variation can be performed

$$\delta\partial_j(\frac{1}{e}u^ku^i)\equiv-\delta\partial_jp^{ik}-\delta p^{s(k}\Gamma^{i)}_{sj}-p^{s(k}\delta\Gamma^{i)}_{sj}-\delta(\frac{1}{e}u^su^{(k})\Gamma^{i)}_{sj}-\frac{1}{e}u^su^{(k}\delta\Gamma^{i)}_{sj}, \tag{42}$$

and therefore, we have no longer a commutation relation for the two operators in this particular equation.

Note that 1. The derivation of the last equation is crucial for proving the noncommutativity property of the variation and the partial differentiation in respect to the projection tensor. The previously derived equations (38) and (39) **cannot** by themselves prove this property. 2. For the moment we have not made use of any assumption, concerning *commutativity* of the partial

differentiation with the variation in respect to the *initially given metric tensor* $g_{\alpha\beta}$. Therefore, the variation of the initial Christoffel connection is assumed to be different from zero, and the same applies also to the last term in (40).

Now, the last expression (42) can be substituted into the first term on the right-hand side of (41) to give

$$g^{ik}\left[\delta,\partial_j\right]p_{mk}+p_{mk}\left[\delta,\partial_j\right]p^{ik}\equiv g_{mk}\partial_j\delta(\frac{1}{e}u^iu^k)+g_{mk}\delta\partial_jp^{ik}+$$

$$+g_{mk}\delta p^{s(k}\Gamma^{i)}_{sj}+g_{mk}p^{s(k}\delta\Gamma^{i)}_{sj}+g_{mk}\delta(\frac{1}{e}u^su^{(k})\Gamma^{i)}_{sj}+$$

$$+g_{mk}(\frac{1}{e}u^su^{(k})\delta\Gamma^{i)}_{sj}-(\frac{1}{e}u^iu^k)\left[\delta,\partial_j\right](\frac{1}{e}u_mu_k). \tag{43}$$

Also, the orthogonality property (37) can be written for the contravariant tensor p^{ik} in the form $\frac{1}{e}u_ku_mp^{ki}\equiv 0$, and the analogous to (38) expression for the commutator is

$$\frac{1}{e}u_ku_m\left[\delta,\partial_j\right]p^{ik}\equiv -p^{ik}\left[\delta,\partial_j\right](\frac{1}{e}u_ku_m). \tag{44}$$

Summing up the last two equations (43) and (44) and contracting the resulting equation with g_{ir}, it can be received, after some transformations:

$$\begin{aligned}
\left[\delta,\partial_j\right]p_{mr} \equiv\ & g_{ir}g_{mk}\partial_j\left[\delta p^{ik}+\delta(\frac{1}{e}u^iu^k)\right]+\partial_j\delta p_{mr}+\\
&+g_{k(r}\left[p^{si}g_{m)i}+\frac{1}{e}u_{m)}u^s\right]\delta\Gamma^k_{sj}+\frac{1}{e}g_{l(r}u_{m)}\Gamma^l_{kj}\delta u^k+\\
&+\Gamma^l_{sj}g_{l(r}\left[\frac{1}{e}u_ku_{m)}\delta^s_n+\frac{1}{e}u^su_ng_{m)k}\right]\delta p^{nk}-\\
&-g_{l(r}g^{np}p_{pm)}\Gamma^l_{sj}g^{sk}\delta p_{nk}-\frac{1}{e^2}\Gamma^l_{nj}g_{l(r}u_{m)}u^nu^k\delta u_k..
\end{aligned} \tag{45}$$

Note that this expression does not contain the commutator $\left[\delta,\partial_j\right]p^{ik}$, since it has cancelled after the summation of (43) and (44). Also, it has been assumed that the variation and the partial derivative commute in respect to the covariant vector field

$$\left[\delta,\partial_j\right]u_m=0. \tag{46}$$

Note, however that in respect to the *contravariant vector field* the following relation is fullfilled

$$[\delta, \partial_j]\, u^m = u_k\, [\delta, \partial_j]\, g^{mk} + g^{mk}\, [\delta, \partial_j]\, u_k. \tag{47}$$

In other words, the variation and the partial derivative commute in respect to the contravariant vector field if and only if they commute in respect to the covariant vector field (equation (46) and also to the metric tensor g_{ik}, i.e. $[\delta, \partial_j]\, g^{mk} = 0$. Relation (45) for the variation $[\delta, \partial_j]\, p_{mr}$ can be additionally simplified if all the variations of contravariant quantities will be found and expressed through the variations of the covariant ones. However, even after that the commutator (45) remains non-zero.

V. DISCUSSION AND CONCLUSIONS.

In this paper a proposition has been used, which states under what conditions the partial derivative and the variation commute with each other - zero-covariant derivative of the tensor field and its form variation, and also zero connection variation. The essence of the non-commutation (or commutation) property can be understood if one tries to realize what is the effect of the two differential operations- the variation and the partial differentiation, when applied in a different order. Let us take first the partial differentiation. According to the standard definition, it is merely the difference between the functional values of the tensor field components at two different spacetime points. However, the functional values are compared after they "resume" the previous (i.e. initial) space-time point (but the two functional values remain of course different). However, in a curved space-time the process of determining the functional value differences is possible only if the *connection variation* from (space-time) point to point is given. Acting subsequently with the (form) variation operator, it is understood that the variation acts also on the connection itself. To put it into another way, this means that the form-variation is determined by the connection variation because the variation is performed *after* partial differentiation - an operation, accounted by the functional tensor components (form) variations, but depending in fact also on the space-time properties. Now, if the form variation is first performed, by the nature of its mathematical definition, *it doesn't account for the presence of a curved space-time due to the simple reason that in no way the connection can enter in the functional variation (unlike the previous case).* In both cases,

when partial differentiation is performed in different order, this differentiation is applied to tensor field components with *different functional values,* which evolve in a different and complicated manner from one space-time point to another. This explains why the different consequitive applications of these operators leads to different results.

Finally, since the "commutation" conditions are evidently not fulfilled for the projection tensor field, in Section IV the exact expression (45) for the commutator has been found on the base of the three defining equations. It is important to mention, however, that in fact we have two expressions for this commutator - the first is the already mentioned eq. (45) and the other one is the derived in Section IV formulae (31), but for the projection tensor

$$\left[\overline{\delta}, \partial_\alpha\right] p_{\mu\nu} \equiv \overline{\delta}(p_{\mu\nu||\alpha}) - (\overline{\delta} p_{\mu\nu})_{|\bar{|}\alpha} - \overline{\delta \Gamma}^r_{\alpha(\mu} p_{\nu)r}, \tag{48}$$

where the double symbol || denotes covariant differentiation with respect to the projection connection $\overline{\Gamma}^r_{\alpha\mu}$, defined in the standard way

$$\overline{\Gamma}^r_{\alpha\mu} \equiv \frac{1}{2} p^{rs}(\partial_\alpha p_{\mu s} + \partial_\mu p_{\alpha s} - \partial_r p_{\alpha\mu}). \tag{49}$$

Note also that the projection connection is determined through a projection tensor field, which does not have an inverse one and this case is different from the standard Christoffell connection.

The last expression (48) is evidently related to performing a covariant differentiation in respect to the projection connection and therefore - to the kinds of transports in a space-time, determined by this connection. However, if one has a classification of the kinds of transports in a space-time with the initial connection, this doesn't mean that such a classification is available also for a space time with the projection connection. That is the reason in the present paper formulae (45) has been used as the more preferred and suitable one and not (48). A problem for further investigation remains open and without an answer yet - whether and under conditions the two expressions can be equivalent, if this is at all possible.

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